Constructing Voter Preferences

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joint work with
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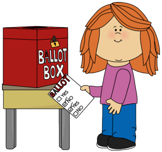
MACALESTER

Separability of Voter Preferences

Motivation

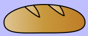
- Multi-criteria decision problem
  - Is a preference on a set of criteria independent of outcomes on other criteria?
  - If not:
    - Encourages strategic voting rather than true preferences
    - No voter's ballot matches the final outcome (Brams et al., 1997)
- Applications in political science, economics, etc.
  - Design and interpretation of referendum elections
  - Bundle purchasing



2-Item Bundle: Hotdog and Bun

2 questions, binary options (✓=Yes, ✗=No)

- Should I buy Hotdogs?
- Should I buy Buns?

		
1	✓	✓
2	✓	✗
3	✗	✗
4	✗	✓

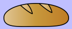
This is a **preference order**

✓✓ $\succ$ ✓✗ $\succ$ ✗✗ $\succ$ ✗✓

$$11 \succ 10 \succ 00 \succ 01$$

$$\{1,2\} \succ \{1\} \succ \emptyset \succ \{2\}$$

Hotdogs and Buns

		
1	✓	✓
2	✓	✗
3	✗	✗
4	✗	✓

Hotdogs and Buns

🌭 is separable

	🌭	🍞
1	✓	✓
4	✗	✓

	🌭	🍞
2	✓	✗
3	✗	✗

	🌭	🍞
1	✓	✓
2	✓	✗
3	✗	✗
4	✗	✓

🍞 is not separable

	🌭	🍞
1	✓	✓
2	✓	✗

	🌭	🍞
3	✗	✗
4	✗	✓

Separability

Definition

$S \subseteq [n]$ is **separable** for a preference order when preferences for outcomes on S are *independent* of the outcomes on $[n] \backslash S$.

Aside: Separable sets in a preference order are similar to independent events in a probability measure. But there are preference orderings that are not induced by probability measures.

(See: Comparative Probability Orders)

...and Beer!

			
1	✓	✓	✓
2	✓	✓	✗
3	✗	✓	✓
4	✓	✗	✓
5	✓	✗	✗
6	✗	✓	✗
7	✗	✗	✓
8	✗	✗	✗

 +  is separable.

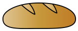
 +  is separable.

AND

 is separable.

...and Beer!

			
1	✓	✓	✓
2	✓	✓	✗
3	✗	✓	✓
4	✓	✗	✓
5	✓	✗	✗
6	✗	✓	✗
7	✗	✗	✓
8	✗	✗	✗

 +  is separable.

		
1	✓	✓
2	✓	✗
3	✗	✓
4	✗	✗

...and Beer!

			
1	✓	✓	✓
2	✓	✓	✗
3	✗	✓	✓
4	✓	✗	✓
5	✓	✗	✗
6	✗	✓	✗
7	✗	✗	✓
8	✗	✗	✗

 +  is separable.

		
1	✓	✓
2	✓	✗
3	✗	✓
4	✗	✗

Closure under intersection

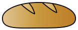
Theorem (Bradley, Hodge, Kilgour, 2005)

If S and T are separable with respect to a preference order, then their intersection $S \cap T$ is also separable.

Warning!

			
1	✓	✓	✓
2	✓	✓	✗
3	✗	✓	✓
4	✓	✗	✓
5	✓	✗	✗
6	✗	✓	✗
7	✗	✗	✓
8	✗	✗	✗

 is separable.

 is separable.

BUT

 is not separable!

Warning

S and T separable **does not** imply that $S \cup T$ is separable.

Warning!

			
1	✓	✓	✓
2	✓	✓	✗
3	✗	✓	✓
4	✓	✗	✓
5	✓	✗	✗
6	✗	✓	✗
7	✗	✗	✓
8	✗	✗	✗

 is separable.

 is separable.

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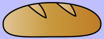
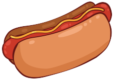
 is not separable!

Warning

S and T separable **does not** imply that $S \cup T$ is separable.

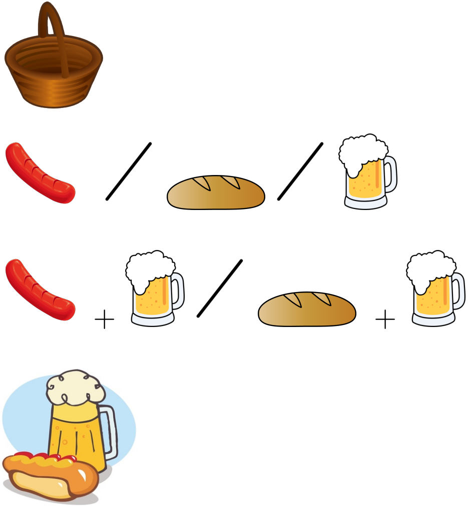
Character of Hotdogs and Buns

		
1	✓	✓
2	✓	✗
3	✗	✗
4	✗	✓



Character of Hotdogs, Buns, and Beer

1	✓	✓	✓
2	✓	✓	✗
3	✗	✓	✓
4	✓	✗	✓
5	✓	✗	✗
6	✗	✓	✗
7	✗	✗	✓
8	✗	✗	✗



Character and Admissibility

Definition

The **character** $\text{char}(\succeq)$ = the collection of subsets $S \subseteq [n]$ that are separable for the preference preorder $\succeq$ (ties are allowed).

- A character always contains $\emptyset$ and $[n]$.
- A character is always closed under intersection.

Definition

A collection $\mathcal{C}$ is **admissible** if there exists a preference preorder $\succeq$ such as $\text{char}(\succeq) = \mathcal{C}$.

Smallest **inadmissible** collection that is closed under intersection:

$$\emptyset, \{1,2\}, \{2\}, \{2,3\}, \{3\}, \{3,4\}, \{1,2,3,4\}$$

The Admissibility Problem

Research Question [Hodge and TerHarr (2008)]

Given a collection $\mathcal{C}$ of subsets of $[n]$ that contains $\emptyset$ and $[n]$ and is closed under intersections. Is there a preference preorder $\succeq$ on $\mathcal{P}([n])$ so that char$(\succeq) = \mathcal{C}$?

Papers on constructing separable characters

- Hodge, Krines and Lahr (2009): completely separable
- Bjorkman, Gravelle, Hodge (2019): Hamilton paths on the hypercube

Overview of Our Results

- We define the **preference space** P^n, which is a 2^n dimensional vector space corresponding to a referendum election with n ballot measures.
- We construct a very useful **voter basis** for P^n
- We use this basis to construct families of admissible characters
  - Collections that are closed under unions and intersections
  - Collections whose Hasse diagram is a tree (excluding $\emptyset$).

Preference Space

Preference Space

The **preference space** P^n is the 2^n-dimensional vector space over $\mathbb{Q}$

A vector $\mathrm{v} \in \mathrm{P}^n$ is called a **preference vector**

$$v = \begin{bmatrix} v(\{1,2\}) \\ v(\{1\}) \\ v(\{2\}) \\ v(\emptyset) \end{bmatrix} = \begin{bmatrix} v(11) \\ v(10) \\ v(01) \\ v(00) \end{bmatrix}$$

and

$$S \succeq T \qquad \text{if and only if} \qquad v(S) \geq v(T).$$

Voter Basis for Preference Space P^n

The **voter basis** for P^n is $\mathcal{B} = \{\mathrm{v}_S : S \subset [n]\}$ where

$$\mathrm{v}_S(X) = \begin{cases} 1 & |S \cap X| \text{ is even} \\ 0 & |S \cap X| \text{ is odd} \end{cases}$$

Pro Tip

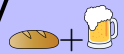
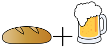
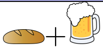
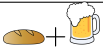
It is helpful to view

S as a **set** and X as an **outcome**.

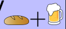
To help us with this distinction we use

- set notation for set S
- binary notation for outcome X

Voter Basis (Cont.)

S	X 	$S \cap X$	$\lvert S \cap X \rvert$	**Parity**	v 
	✓✓✓		2	Even	1
	✓✓✗		1	Odd	0
	✓✗✓		1	Odd	0
	✓✗✗		0	Even	1
	✗✓✓		2	Even	1
	✗✓✗		1	Odd	0
	✗✗✓		1	Odd	0
	✗✗✗		0	Even	1

Voter Basis for Preference Space P^3

	v 	v 	v 	v 	v 	v 	v 	v 
✓✓✓	0	1	1	0	1	0	0	1
✓✓✗	1	1	0	0	0	0	1	1
✓✗✓	1	0	1	0	0	1	0	1
✓✗✗	0	0	0	0	1	1	1	1
✗✓✓	1	0	0	1	1	0	0	1
✗✓✗	0	0	1	1	0	0	1	1
✗✗✓	0	1	0	1	0	1	0	1
✗✗✗	1	1	1	1	1	1	1	1

Preference Preorders Induced by Voter Basis Vectors

$\mathrm{v}_S \in \mathcal{B}$ induces a preference preorder $\succeq$ in which outcomes that are **even** in S are preferred over outcomes that are **odd** in S.

v_S	induced preference order
$\mathrm{v}_{\{1,2,3\}}$	$\{110, 101, 011, 000\} \succ \{001, 010, 100, 111\}$
$\mathrm{v}_{\{1,2\}}$	$\{110, 111, 001, 000\} \succ \{101, 100, 011, 010\}$
$\mathrm{v}_{\{1\}}$	$\{011, 010, 001, 000\} \succ \{111, 110, 101, 100\}$
$\mathrm{v}_{\emptyset}$	$\{111, 110, 101, 100, 011, 010, 001, 000\}$

We can build any preference preorder via linear combinations of the v_S.

Voter Basis for Preference Space P^n

Theorem (B, Calaway 2021)

The vectors $\mathrm{v}_S(X) = \begin{cases} 1 & |S \cap X| \text{ is even,} \\ 0 & |S \cap X| \text{ is odd,} \end{cases}$ form a basis for P^n.

Note: Variation of Young's seminormal representation of $\mathbb{Z}_2 \wr S_n$.

Lemma (B, Calaway 2021)

The preference preorder induced by the voter vector $\mathrm{v}_S(X)$ is separable on X if and only if

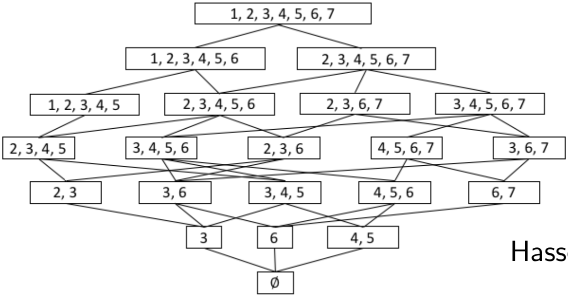
$$S \subset X \quad \text{or} \quad S \cap X = \emptyset.$$

Admissible Character Construction

Distributive Subset Lattice

Definition

A *distributive subset lattice* $\mathcal{L} = \{L_1, L_2, \ldots, L_m\} \subset \mathcal{P}([n])$ is a collection that contains $\emptyset$ and $[n]$ and is closed under intersections and unions.



Hasse Diagram

Distributive Subset Lattices are Admissible

Theorem (B., Nguyen, Tran, 2021+)

Every distributive subset lattice $\mathcal{L} \subset \mathcal{P}([n])$ is admissible.

Proof: We use the voter basis

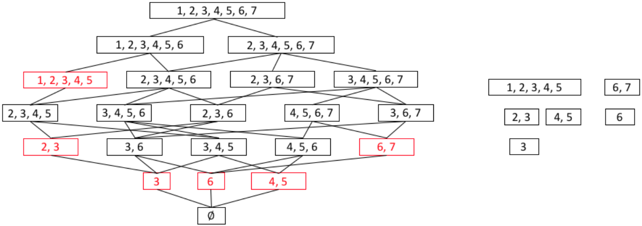
$$\{\mathrm{v}_S \mid S \subset [n]\}$$

of $\mathrm{P}^n \cong \mathbb{Q}^{2n}$ to construct a preference vector (i.e. utility function) with character $\mathcal{L}$.

We will try to use as few basis elements as possible.

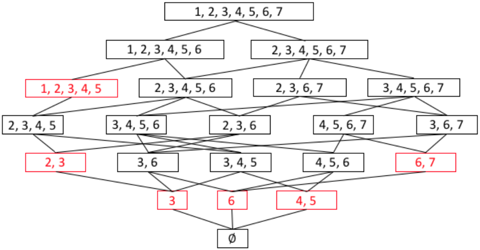
Join Irreducibles of $\mathcal{L}$

What is the smallest subcollection $\mathcal{A} \subset \mathcal{L}$ that we can use to generate $\mathcal{L}$ via set unions?



Use the **join irreducible** members of the subset lattice $\mathcal{L}$.
(X is join irreducible when $X = Y \cup Z$ forces $X = Y$ or $X = Z$.)

Distributive Subset Lattices are Admissible



$$v = v_{12345} + 2v_{67} + 4v_{45} + 8v_{23} + 16v_6 + 32v_3$$

- The entries of v describe a utility function on the outcomes.
- This utility function induces a preference preorder $\succeq$.
- The character (collection of separable sets) of the preference preorder is $\mathcal{L}$.

Distributive Subset Lattices are Admissible

Theorem (B., Nguyen, Tran, 2021+)

Let $\mathcal{L} = \{L_1, L_2, \ldots, L_m\}$ be a distributive character with join irreducible elements $\mathcal{A} = \{A_1, A_2, \ldots, A_\ell\} \subset \mathcal{L}$. Then $\mathcal{L}$ is the character induced by the preference vector

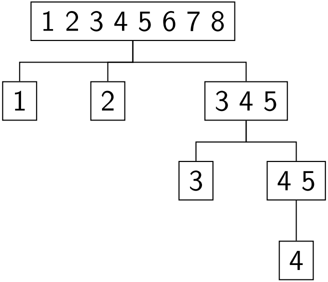
$$\mathsf{v} = \sum_{i=1}^{\ell} 2^{\rho(A_i)} \mathsf{v}_{A_i}$$

where $\rho(A_i)$ is the rank of set A_i in the reverse lexicographical ordering.

Tree Collection

Definition

A *tree collection* is a family of subsets $\mathcal{T} \subset \mathcal{P}([n])$ such that $\emptyset, [n] \in \mathcal{T}$, and every pair of sets $A_i, A_j \in \mathcal{T}$ is either nested or disjoint.



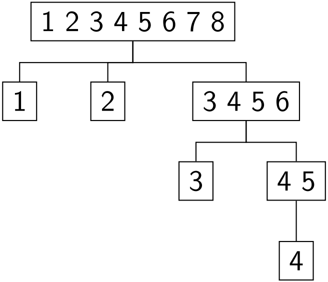
The Hasse diagram of $\mathcal{T} \backslash \{\emptyset\}$ is a tree.

Tree Collections are Admissible

Theorem (B., Calaway 2021)

Every tree collection $\mathcal{T} \subset \mathcal{P}([n])$ is admissible.

Desired Tree Collection $\mathcal{T}$



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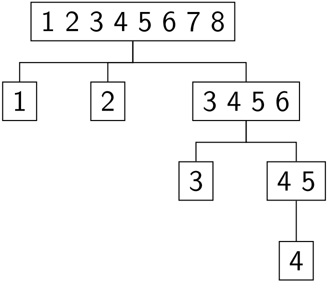

order induced by preference vector

$$v_{\mathcal{T}} = \sum_{A \in \mathcal{T}} c_A \mathbf{v}_A$$

Tree Collections are Admissible

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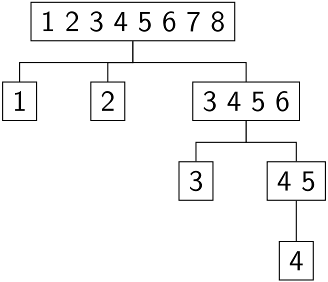
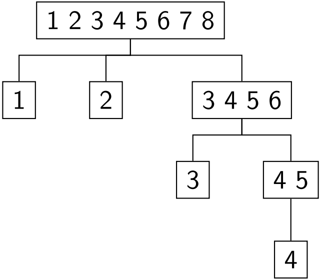


order induced by preference vector

$$v_{\mathcal{T}} = \sum_{A \in \mathcal{T}} c_A v_A + \sum_{B \in \mathcal{U}} d_B v_B$$

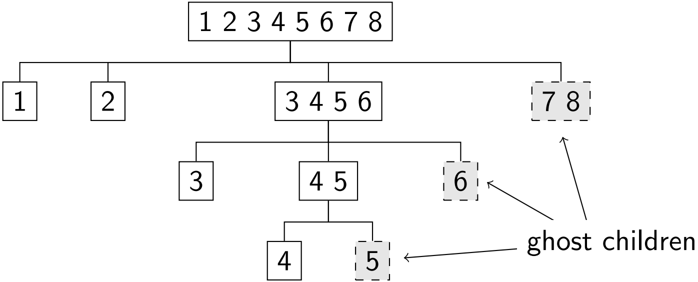
small noise breaks unwanted unions

Breaking Separability of Unions of Siblings



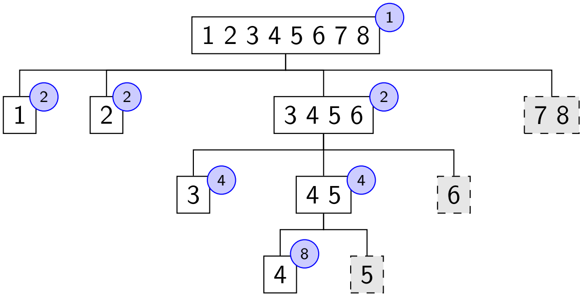
Hasse Diagram

Breaking Separability of Unions of Siblings



Haunted Hasse Diagram

Algorithm for Tree Collection Construction



Then put small, distinct weights on unions of siblings

12 13456 178 2345 278 345678 345 36 456 45

□

Conclusion and Future Work

- Distributive Lattice Collections are admissible.
- Tree Collections are admissible.

Research Question

What other families of separable characters can we build with the voter basis?

Completely Separable Preference Orders

The number of non-isomorphic completely separable preference orders, i.e. where every set is separable.

# proposals	# non-isomorphic completely separable preferences
1	1
2	1
3	2
4	14
5	546
6	169,444

Research Question

Can we use the voter basis to shed light on the structure of completely separable preferences? To construct them?

The Big Prize: Completely Separable Preferences

The 14 completely separable preference orders with respect to the voter basis.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}
$v_\emptyset$	64	64	64	64	64	64	64	64	64	64	64	64	64	64
$v_{\{1\}}$	−4	−1	−5	−4	−10	−7	−6	−3	−7	−6	−9	−12	1	−2
$v_{\{2\}}$	−8	−9	−9	−12	−6	−3	−14	−11	−11	−10	−5	−8	−5	−8
$v_{\{3\}}$	−16	−17	−17	−20	−20	−17	−18	−15	−15	−14	−15	−18	−15	−18
$v_{\{4\}}$	−32	−31	−31	−28	−28	−31	−28	−31	−31	−32	−31	−28	−31	−28
$v_{\{1,2\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{\{1,3\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{\{1,4\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{\{2,3\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{\{2,4\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{\{3,4\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$v_{\{1,2,3\}}$	0	3	−1	0	0	3	2	5	1	2	5	2	−5	−8
$v_{\{1,2,4\}}$	0	−3	1	0	0	−3	0	−3	1	0	−3	0	7	10
$v_{\{1,3,4\}}$	0	−3	1	0	6	3	0	−3	1	0	3	6	−7	−4
$v_{\{2,3,4\}}$	0	1	1	4	−2	−5	4	1	1	0	−5	−2	−5	−2
$v_{\{1,2,3,4\}}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Converting a Preorder into an Order

The official definition of admissibility requires $\succeq$ to be an order (rather than a preorder).

Lemma

Let $\mathrm{v} \in \mathrm{P}^n$ induce preference preorder $\succeq$ with character $\mathcal{C}$. Suppose that for every nonseparable set $B \notin \mathcal{C}$, there exist partial outcomes $x_B, y_B, u_{[n]-B}, v_{[n]-B}$ such that

$$x_B\, u_{[n]-B} \succ y_B\, u_{[n]-B} \quad \text{and} \quad x_B\, v_{[n]-B} \prec y_B\, v_{[n]-B}.$$

Then there is preference vector v' that induces a preference order $\succeq'$ with character $\mathcal{C}$.

Technical Condition: For every nonseparable B, a some partial preference must truly flip (rather than changing to indifference).