# de Finetti Lattices and Magog Triangles

Andrew Beveridge[1]

joint work with
Ian Calaway[2] and Kristin Heysse[1]

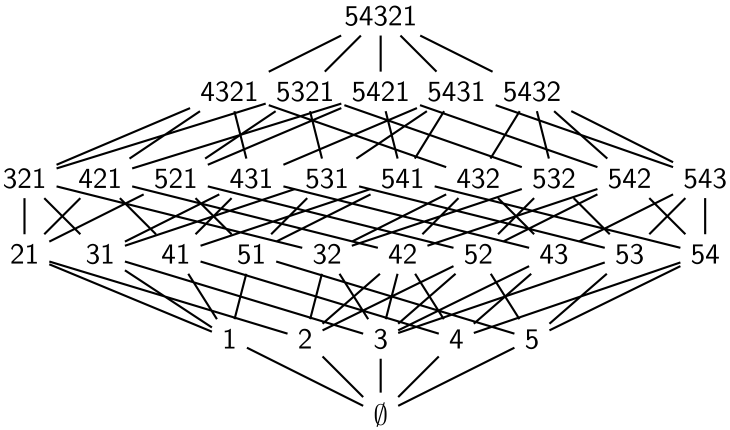
Macalester College[1] and Stanford University[2]

Canadian Discrete and Algorithmic Mathematics Conference
May 2021

MACALESTER

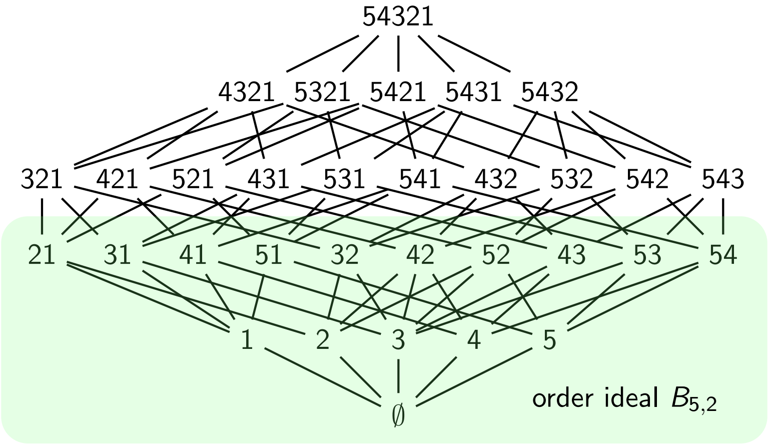
# Introduction

# Boolean Lattice $B_n$ and Order Ideal $B_{n,2}$



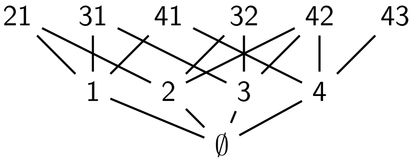

$B_5 =$ subsets of $\{1, 2, 3, 4, 5\}$ ordered by inclusion

# Boolean Lattice $B_n$ and Order Ideal $B_{n,2}$

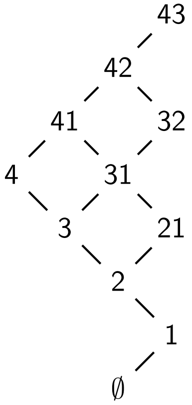



$B_5$ = subsets of $\{1, 2, 3, 4, 5\}$ ordered by inclusion

# de Finetti Lattice $F_{n,2}$




order ideal $B_{4,2}$
of Boolean Lattice $B_4$



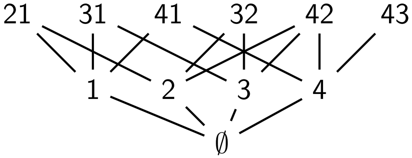

de Finetti Lattice $F_{4,2}$

# de Finetti Lattice $F_{n,2}$

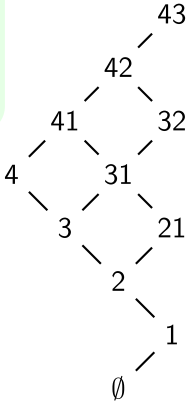
$$\{i\} \prec \{i,j\}$$

$$\{i\} \prec \{j\} \longleftrightarrow i < j$$

$$\{i,j\} \preceq \{k,\ell\} \longleftrightarrow i \leq k \text{ and } j \leq \ell$$




order ideal $B_{4,2}$
of Boolean Lattice $B_4$




de Finetti Lattice $F_{4,2}$

# Motivation for de Finetti Lattice $F_{n,2}$

## Definition

A *de Finetti refinement* $(F, \prec_F)$ of the Boolean lattice $(B_n, \prec)$ is a poset refinement that adheres to

(F1) $\emptyset \prec \{1\} \prec \{2\} \prec \ldots \prec \{n\}$, and

(F2) $X \prec Y$ if and only if $X \cup Z \prec Y \cup Z$ for all $Z \subset [n]$ such that $(X \cup Y) \cap Z = \emptyset$.

for all sets $X, Y \subset [n]$ that are comparable in $F$.

A *de Finetti total order* is a linear extension of $B_n$ that adheres to (F1) and (F2).

example: $3 \prec 21 \quad \longleftrightarrow \quad 543 \prec 5421$

Condition (F2) is de Finetti's axiom (1931)

# de Finetti Total Orders

de Finetti total orders of $B_n$ appear in various settings:

- Comparable Probability Orders (probability)
- Boolean Term Orders (computational algebra)
- Completely Separable Preferences (social choice theory)

**Open Question**: General formula for the number of de Finetti total orders of $B_n$ (OEIS A005806)

$$1 \quad 1 \quad 2 \quad 14 \quad 546 \quad 169{,}444 \quad 560{,}043{,}206$$

(The first 14 terms are known.)

# Our Results for Refinements of $F_{n,2}$

**Theorem [B, Calaway, Heysse (2021)]**

The collection $\mathcal{F}_{n,2}$ of linear extensions of $F_{n,2}$ are in bijection with Strict Sense Ballots for $n$ candidates.

strict sense ballots $\longleftrightarrow$ shifted standard Young tableaux of shape $(n, n-1, \ldots .1)$

**Theorem [B, Calaway, Heysse (2021)]**

The collection $\mathcal{F}^1_{n,2}$ of poset refinements of $F_{n,2}$ that are induced by resolving all disjoint pairs $\{i\}, \{k, \ell\}$ are in bijection with Magog Triangles of size $n-1$.

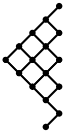
magog triangles $\longleftrightarrow$ totally symmetric self-complementary plane partitions

# Linear Extensions of $F_{n,2}$

# Linear Extensions of $F_{n,2}$

**Theorem [B, Calaway, Heysse (2021)]**

The collection $\mathcal{F}_{n,2}$ of linear extensions of $F_{n,2}$ are in bijection with Strict Sense Ballots for $n$ candidates.



The strict-sense ballot number sequence (OEIS A003121) begins with

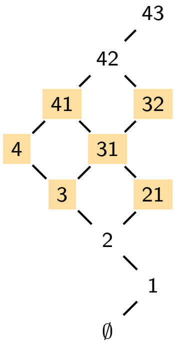
$$1 \quad 1 \quad 2 \quad 12 \quad 286 \quad 33,592 \quad 23,178,480 \quad \ldots$$

and the general formula for the $n$th strict-sense ballot number is

$$\binom{n+1}{2}! \; \frac{\prod_{k=1}^{n-1} k!}{\prod_{k=1}^{n} (2k-1)!}.$$

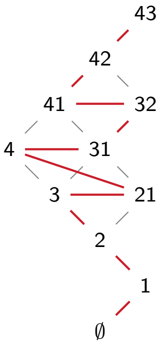
# Proof Part 1: Linear Extension to Shifted SYT

Linear Extension of $F_{n,2}$ $\iff$ Shifted Standard Young Tableau of shape $(n, n-1, \ldots, 1)$.



# Proof Part 1: Linear Extension to Shifted SYT

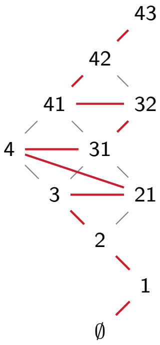
Linear Extension of $F_{n,2}$ $\iff$ Shifted Standard Young Tableau of shape $(n, n-1, \ldots, 1)$.

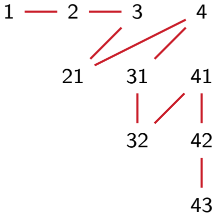


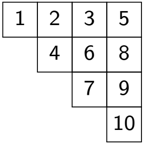

pick a valid total order

# Proof Part 1: Linear Extension to Shifted SYT

Linear Extension of $F_{n,2}$ $\Longleftrightarrow$ Shifted Standard Young Tableau of shape $(n, n-1, \dots, 1)$.



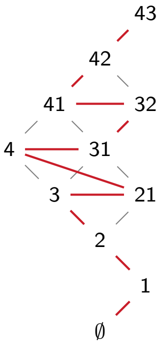

pick a valid total order

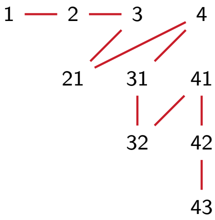
rotate and omit ∅

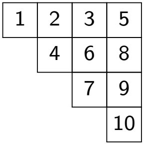
record order visited

# Proof Part 1: Linear Extension to Shifted SYT

Linear Extension of $F_{n,2}$ $\iff$ Shifted Standard Young Tableau of shape $(n, n-1, \ldots, 1)$.




pick a valid total order

rotate and omit ∅

record order visited

de Finetti refinement $\iff$ rows and columns increase

# Proof Part 2: Shifted SYT to Strict Sense Ballot (folklore)

Strict Sense Ballot with $n$ candidates:

- Candidate $k$ receives exactly $n+1-k$ votes for $1 \le k \le n$
- During the vote count, candidate $k$ always strictly ahead of candidate $\ell$ for all $1 \le k < \ell \le n$

# Proof Part 2: Shifted SYT to Strict Sense Ballot (folklore)

Strict Sense Ballot with $n$ candidates:

- Candidate $k$ receives exactly $n+1-k$ votes for $1 \leq k \leq n$
- During the vote count, candidate $k$ always strictly ahead of candidate $\ell$ for all $1 \leq k < \ell \leq n$

Shifted SYT $\iff$ Strict Sense Ballot

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 5 |
| | 4 | 6 | 8 |
| | | 7 | 9 |
| | | | 10 |

# Proof Part 2: Shifted SYT to Strict Sense Ballot (folklore)

Strict Sense Ballot with $n$ candidates:

- Candidate $k$ receives exactly $n+1-k$ votes for $1 \leq k \leq n$
- During the vote count, candidate $k$ always strictly ahead of candidate $\ell$ for all $1 \leq k < \ell \leq n$

Shifted SYT $\Longleftrightarrow$ Strict Sense Ballot

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 5 |
| | 4 | 6 | 8 |
| | | 7 | 9 |
| | | | 10 |

$\Longleftrightarrow$ 1, 1, 1, 2, 1, 2, 3, 2, 3, 4

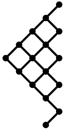
Row $i$ gives the indices of the votes for candidate $i$. □

# Singleton Refinements of $F_{n,2}$

# Singleton Refinements of $F_{4,2}$

**Theorem [B, Calaway, Heysse (2021)]**

The collection $\mathcal{F}^1_{n,2}$ of poset refinements of $F_{n,2}$ that are induced by resolving all disjoint pairs $\{i\}, \{k, \ell\}$ are in bijection with Magog Triangles of size $n-1$.



The magog triangle sequence (OEIS A005130) begins with

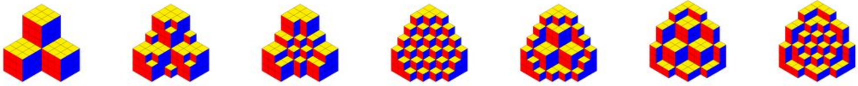
$$1 \quad 2 \quad 7 \quad 42 \quad 429 \quad 7,436 \quad 218,348 \quad \ldots$$

and the general formula is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

# TSSCPP are in bijection with Magog Triangles

**Totally Symmetric Self-Complementary Plane Partition (TSSCPP)**: A stack of cubes in a $2n \times 2n \times 2n$ box with the maximum possible symmetry.



| | | |
|---|---|---|
| 1 | | |
| 1 | 1 | |
| 1 | 1 | 1 |

| | | |
|---|---|---|
| 1 | | |
| 1 | 1 | |
| 1 | 1 | 2 |

| | | |
|---|---|---|
| 1 | | |
| 1 | 1 | |
| 1 | 1 | 3 |

| | | |
|---|---|---|
| 1 | | |
| 1 | 1 | |
| 1 | 2 | 3 |

| | | |
|---|---|---|
| 1 | | |
| 1 | 1 | |
| 1 | 2 | 2 |

| | | |
|---|---|---|
| 1 | | |
| 1 | 2 | |
| 1 | 2 | 2 |

| | | |
|---|---|---|
| 1 | | |
| 1 | 2 | |
| 1 | 2 | 3 |

**Magog Triangle**: triangular array $M(i,j)$ where $1 \leq j \leq i \leq n$ and $1 \leq M(i,j) \leq j$ with weakly increasing columns and weakly increasing rows.

Image adapted from Fischer and Konvalinka, PNAS (2020)

# TSSCPP and ASM and DPP and AST are equinumerous

Four distinct families enumerated by $\displaystyle\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$

Alternating Sign Matrices

Descending Plane Partitions

TSSCPP

Alternating Sign Triangles

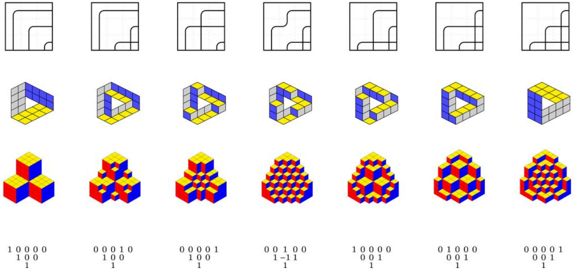
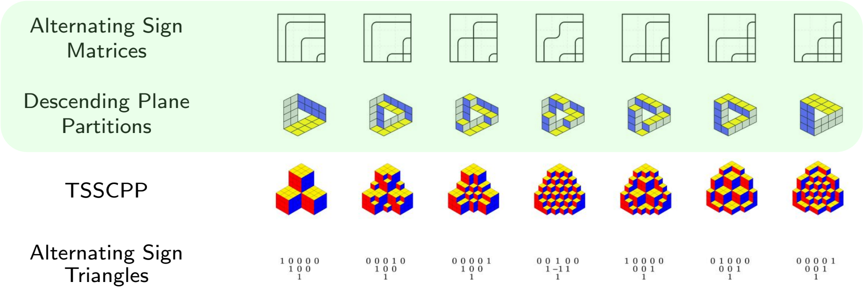



Image adapted from Fischer and Konvalinka, PNAS (2020)

# TSSCPP and ASM and DPP and AST are equinumerous

~~Four~~ Three distinct families enumerated by $\displaystyle\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}$




Breakthrough: Fischer and Konvalinka (2019) found a bijection between ASMs and DPP!

Image adapted from Fischer and Konvalinka, PNAS (2020)

# Magog Triangles are in bijection with Kagog Triangles

magog triangle $M(i,j)$

- $n$ rows
- $1 \leq M(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

| | | | | | |
|---|---|---|---|---|---|
| 1 | | | | | |
| 1 | 1 | | | | |
| 1 | 1 | 1 | | | |
| 1 | 1 | 2 | 2 | | |
| 1 | 2 | 2 | 2 | 3 | |
| 1 | 2 | 2 | 4 | 5 | 5 |

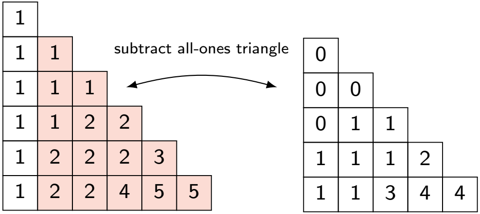
# Magog Triangles are in bijection with Kagog Triangles

magog triangle $M(i,j)$

- $n$ rows
- $1 \leq M(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

omagog triangle $M^{\circ}(i,j)$

- $n-1$ rows
- $0 \leq M^{\circ}(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing



# Magog Triangles are in bijection with Kagog Triangles

magog triangle $M(i,j)$

- $n$ rows
- $1 \leq M(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

omagog triangle $M^{\circ}(i,j)$

- $n-1$ rows
- $0 \leq M^{\circ}(i,j) \leq j$
- columns weakly increasing
- rows weakly increasing

NEW!

kagog triangle $K(i,j)$

- $n-1$ rows
- $0 \leq K(i,j) \leq j$
- columns weakly decreasing
- rows can start with multiple zeros, then positive entries strictly increasing

| | | | | | |
|---|---|---|---|---|---|
| 1 | | | | | |
| 1 | 1 | | | | |
| 1 | 1 | 1 | | | |
| 1 | 1 | 2 | 2 | | |
| 1 | 2 | 2 | 2 | 3 | |
| 1 | 2 | 2 | 4 | 5 | 5 |

subtract all-ones triangle ↔

| | | | | |
|---|---|---|---|---|
| 0 | | | | |
| 0 | 0 | | | |
| 0 | 1 | 1 | | |
| 1 | 1 | 1 | 2 | |
| 1 | 1 | 3 | 4 | 4 |

two-color cube pyramid ↔

| | | | | |
|---|---|---|---|---|
| 1 | | | | |
| 0 | 1 | | | |
| 0 | 1 | 2 | | |
| 0 | 0 | 2 | 4 | |
| 0 | 0 | 0 | 1 | 3 |

# Magog Triangles are in bijection with Kagog Triangles

**Theorem [B, Calaway, Heysse (2021)]**

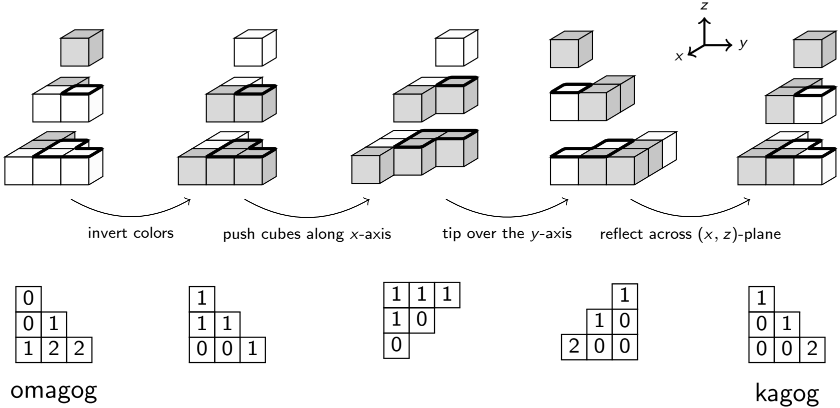
Kagog triangles are in bijection with omagog triangles (and therefore with magog triangles and TSSCPPs).

**Proof Sketch:**

- Represent an omagog triangle as a **two-color cube pyramid**
  - White cubes are present
  - Gray cubes are missing
- Invert the colors
- Perform a series of elementary geometric transformations
- The result is a kagog pyramid

□

# Example of omagog to kagog mapping

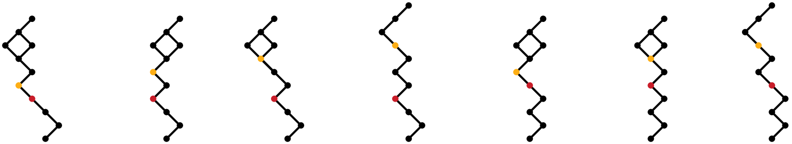



The kagog triangle is made from the “missing cubes” of the omagog triangle.

# Back to Singleton Refinements of $F_{4,2}$

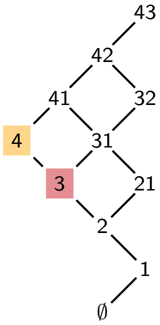
**Theorem [B, Calaway, Heysse (2021)]**

The collection $\mathcal{F}^1_{n,2}$ of poset refinements of $F_{n,2}$ that are induced by resolving all disjoint pairs $\{i\}, \{k, \ell\}$ are in bijection with ~~Magog Triangles of size $n-1$~~ Kagog Triangles of size $n-2$.

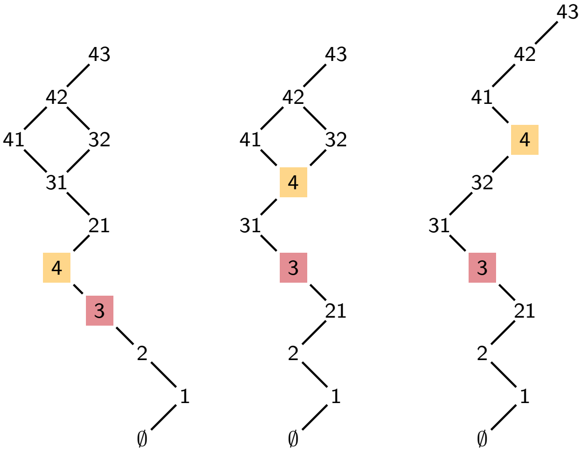


| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| 1 2 | 0 2 | 0 1 | 0 0 | 0 2 | 0 1 | 0 0 |

# Singleton Refinements of $F_{4,2}$

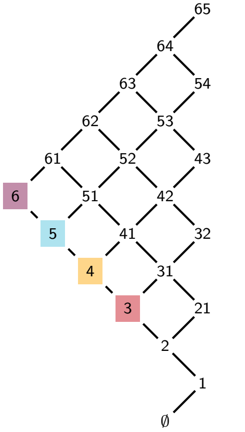



de Finetti Lattice $F_{4,2}$




three different refinements (out of seven)

# Proof By Example: Refining $F_{6,2}$ and Creating Kagog




de Finetti Lattice $F_{6,2}$

3

| | |
|---|---|
| 21 | |

4

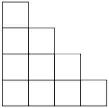
| | |
|---|---|
| 31 | 32 |
| 21 | |

5

| | | |
|---|---|---|
| 41 | 42 | 43 |
| 31 | 32 | |
| 21 | | |

6

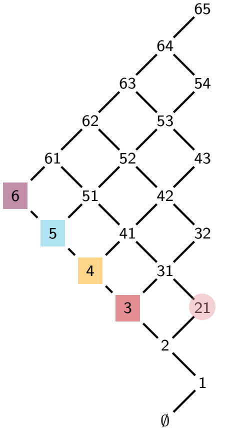
| | | | |
|---|---|---|---|
| 51 | 52 | 53 | 54 |
| 41 | 42 | 43 | |
| 31 | 32 | | |
| 21 | | | |



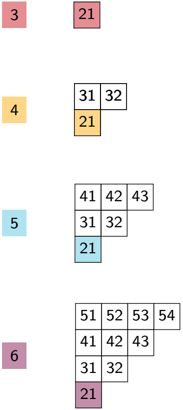
kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

# Proof By Example: Refining $F_{6,2}$ and Creating Kagog

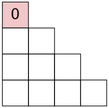



de Finetti Lattice $F_{6,2}$




$21 \prec 3$

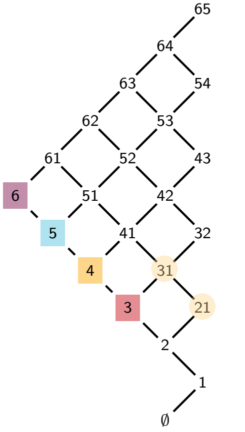
*count white boxes in each row*



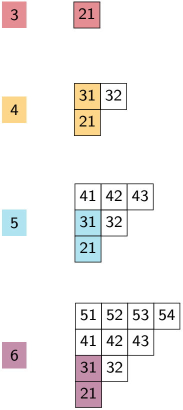

kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

# Proof By Example: Refining $F_{6,2}$ and Creating Kagog



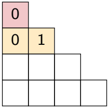

de Finetti Lattice $F_{6,2}$




$21 \prec 3$

$31 \prec 4 \prec 32$

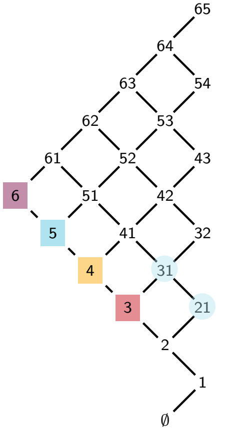
*count white boxes in each row*



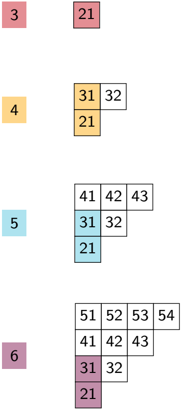

kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

# Proof By Example: Refining $F_{6,2}$ and Creating Kagog




de Finetti Lattice $F_{6,2}$

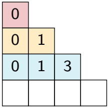



$21 \prec 3$

$31 \prec 4 \prec 32$

$31 \prec 5 \prec 32$
$5 \prec 41$

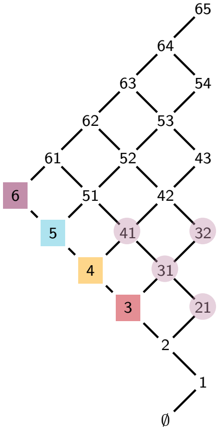
*count white boxes in each row*




kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

# Proof By Example: Refining $F_{6,2}$ and Creating Kagog




de Finetti Lattice $F_{6,2}$

**3**

| | |
|---|---|
| 21 | |

$21 \prec 3$

**4**

| | |
|---|---|
| 31 | 32 |
| 21 | |

$31 \prec 4 \prec 32$

**5**

| | | |
|---|---|---|
| 41 | 42 | 43 |
| 31 | 32 | |
| 21 | | |

$31 \prec 5 \prec 32$
$5 \prec 41$

**6**

| | | | |
|---|---|---|---|
| 51 | 52 | 53 | 54 |
| 41 | 42 | 43 | |
| 31 | 32 | | |
| 21 | | | |

$32 \prec 6$
$41 \prec 6 \prec 42$
$6 \prec 51$

*count white boxes in each row*

| | | | |
|---|---|---|---|
| 0 | | | |
| 0 | 1 | | |
| 0 | 1 | 3 | |
| 0 | 0 | 2 | 4 |

kagog triangle

- columns: weakly decreasing
- rows: positive entries strictly increasing

The de Finetti constraints correspond to the kagog constraints. □

# Summary and Future Work

Refinements and linear extensions of de Finetti Lattice $F_{n,2}$

**Our Bijective Results**

| | | |
|---|---|---|
| Linear extensions of $F_{n,2}$ | $\longleftrightarrow$ | Strict Sense Ballots |
| Singleton refinements of $F_{n,2}$ | $\longleftrightarrow$ | Kagog Triangles |
| Kagog Triangles | $\longleftrightarrow$ | Magog Triangles |

Future Directions: find enumerative formulas for

- de Finetti refinements of $F_{n,k}$ for $3 \leq k \leq n$
- de Finetti total orders of $F_{n,n}$ = de Finetti total orders of $B_n$ (aka comparative probability orders, OEIS A005806)

# Some References


- G. Andrews. Plane partitions V: the TSSCPP conjecture. *J. Combin. Theory, Ser. A*, 66:28–39, 1994.
- W. J. Bradley, J. K. Hodge, and D. M. Kilgour. Separable discrete preferences. *Math. Social Sci.*, 49(1):335–353, 2005.
- R. Christian, M. Conder, and A. Slinko. Flippable pairs and subset comparisons in comparative probability orderings. *Order*, 24:193–213, 2007.
- B. de Finetti. Sul significato soggettivo della probabilità. *Fundamenta Mathematicae*, 17:298–329, 1931.
- T. Fine and J. Gill. The enumeration of comparative probability relations. *Ann. Prob.*, 4:667–673, 1976.
- I. Fischer and M. Konvalinka, A bijective proof of the ASM theorem, part II: ASM enumeration and ASM-DPP relation. *Elec. J. Combin.*, 27 (3): #3.35, 2020.
- P. C. Fishburn. Finite linear qualitative probability. *J. Math. Psychology*, 40:64–77, 1996.
- J. K. Hodge. The mathematics of referendums and separable preferences. *Math. Mag.*, 84(4):268–277, 2011.
- C. H. Kraft, J. W. Pratt, and A. Seidenberg. Intuitive probability on finite sets. *Ann. Math. Statist.*, 30:408–419, 1959.
- G. Kuperberg. Another proof of the alternating sign matrix conjecture. *Int. Math. Research Notes*, 1996:139–150, 1996.
- D. Maclagan. Boolean term orders and the root system $b_n$. *Order*, 15:279–295, 1999.
- W.H. Mills, D. P. Robbins, and H. Rumsey. Alternating-sign matrices and descending plane partitions. *J. Combin. Theory, Ser. A*, 34:340–359, 1983.
- A. Slinko. Additive representability of finite measurement structures. In S. J. Brams, W. V. Gehrlein, and F. S. Roberts, editors, *The Mathematics of Preference, Choice and Order*, pages 113–133. Springer, 2009.
- D. Zeilberger. Proof of the alternating sign matrix conjecture. *Elec. J. Comb.*, 3, R13, 1996.

# Alternating Sign Matrices and Gog Triangles

## Alternating Sign Matrices

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&-1&1\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

## Column Sum Matrices

$$\begin{bmatrix}1&0&0\\1&1&0\\1&1&1\end{bmatrix} \quad \begin{bmatrix}1&0&0\\1&0&1\\1&1&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&1&0\\1&1&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\1&0&1\\1&1&1\end{bmatrix} \quad \begin{bmatrix}0&1&0\\0&1&1\\1&1&1\end{bmatrix} \quad \begin{bmatrix}0&0&1\\1&0&1\\1&1&1\end{bmatrix} \quad \begin{bmatrix}0&0&1\\0&1&1\\1&1&1\end{bmatrix}$$

## Gog Triangles (aka Monotone Triangles)

$$\begin{array}{|l|l|l|}\hline 1 \\ \hline 1&2 \\ \hline 1&2&3 \\ \hline\end{array} \quad \begin{array}{|l|l|l|}\hline 1 \\ \hline 1&3 \\ \hline 1&2&3 \\ \hline\end{array} \quad \begin{array}{|l|l|l|}\hline 2 \\ \hline 1&2 \\ \hline 1&2&3 \\ \hline\end{array} \quad \begin{array}{|l|l|l|}\hline 2 \\ \hline 1&3 \\ \hline 1&2&3 \\ \hline\end{array} \quad \begin{array}{|l|l|l|}\hline 2 \\ \hline 2&3 \\ \hline 1&2&3 \\ \hline\end{array} \quad \begin{array}{|l|l|l|}\hline 3 \\ \hline 1&3 \\ \hline 1&2&3 \\ \hline\end{array} \quad \begin{array}{|l|l|l|}\hline 3 \\ \hline 2&3 \\ \hline 1&2&3 \\ \hline\end{array}$$

## Ogog Triangles (subtract minimal Gog Triangle)

$$\begin{array}{|l|l|}\hline 0 \\ \hline 0&0 \\ \hline\end{array} \quad \begin{array}{|l|l|}\hline 0 \\ \hline 0&1 \\ \hline\end{array} \quad \begin{array}{|l|l|}\hline 1 \\ \hline 0&0 \\ \hline\end{array} \quad \begin{array}{|l|l|}\hline 1 \\ \hline 0&1 \\ \hline\end{array} \quad \begin{array}{|l|l|}\hline 1 \\ \hline 1&1 \\ \hline\end{array} \quad \begin{array}{|l|l|}\hline 2 \\ \hline 0&1 \\ \hline\end{array} \quad \begin{array}{|l|l|}\hline 2 \\ \hline 1&1 \\ \hline\end{array}$$

# Two-Color Cube Pyramid Involution for Gog Triangles

**Theorem [B, Calaway, Heysse (2021)]**

There is a gog triangle involution $\phi$ that corresponds to both:

- an affine transformation of two-color cube pyramids, and
- reversing the order of the rows of the corresponding alternating sign matrix.