Approval Ballot Triangles

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joint work with
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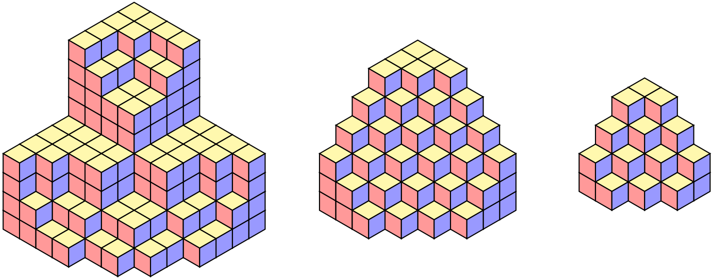
Joint Mathematics Meetings
April 2022

Cold Open



$$= (1,1,1,2,2,3,1,2,3,1,2,4,3,4,5)$$

Introduction

Bertrand's Ballot Problem (1887*)

In a two candidate election,

- Candidate A receives a votes.
- Candidate B receives $b < a$ votes.

There are

$$\frac{a-b}{a+b}\binom{a+b}{a}$$

orderings of the ballots so that A is always ahead of B during the vote count.

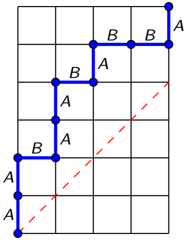
Joseph Bertrand

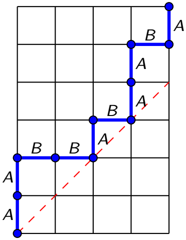
William Allen Whitworth

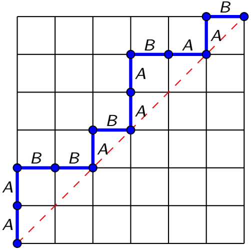
*Fun Fact: Bertrand actually rediscovered William Allen Whitworth's 1878 result.

Ballot Problems as Lattice Paths

$a > b$, no ties allowed | $a > b$, ties allowed | $a = b$, ties allowed







$$\frac{a-b}{a+b}\binom{a+b}{a}$$

$$\frac{a+1-b}{a+b}\binom{a+b}{a}$$

$$C_a = \frac{1}{a+1}\binom{2a}{a}$$

Ballot Sequences: ℓ never trails $\ell + 1$

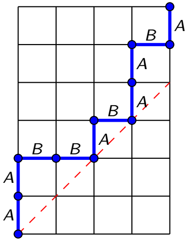
Definition

The sequence $b_1, \ldots, b_n$ where $1 \leq b_k \leq k$ is a **ballot sequence** when every partial sequence $b_1, \ldots, b_k$ contains at least as many ℓ's as $(\ell + 1)$'s for all $1 \leq \ell < k$,

examples	non-examples	
$1, 1, 1$	$1, 2, 2$	too many 2's
$1, 1, 2$	$1, 1, 3$	no 2 before the 3
$1, 2, 1$	$2, 1, 3$	must start with 1
$1, 2, 3$	$1, 3, 2$	no 2 before the 3

In a ballot sequence, the final tally for ℓ is greater than or equal to the final tally for $\ell + 1$.

Ballot Sequences generalize Ballot Problems



$A, \quad A, \quad B, \quad B, \quad A, \quad B, \quad A, \quad A, \quad B, \quad A$

$1, \quad 1, \quad 2, \quad 2, \quad 1, \quad 2, \quad 1, \quad 1, \quad 2, \quad 1$

Bertrand's Ballot Problem is a ballot sequence $b_1, b_2, \ldots, b_n$ where $b_k \in \{1, 2\}$ for $1 \leq k \leq n$.

A Voting Procedure that Creates a Ballot Sequence

Here is a voting procedure that creates a sequence $b_1, b_2, \ldots, b_n$ such that $b_k \in [k]$.

- People enter a room, one at at item.
- Person k casts a ballot for any of the k people **currently in the room**.

The ballots $b_1, b_2, \ldots, b_n$ are a **ballot sequence** provided that "person ℓ never trails person $\ell + 1$" as the votes are cast.

Ballot Sequences are Counted by the Involution Numbers

Ballot sequences of length n are in bijection with standard Young tableaux (SYT) of size n.

- b_k records the SYT **row** that contains element k

$$(1,1,2,1,3,2,1,2) \longleftrightarrow \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 7 \\ \hline 3 & 6 & 8 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array} \longleftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 5 & 3 & 6 & 2 & 4 & 8 & 7 \end{pmatrix}$$

SYT of size n are in bijection with involutions of $[n]$ via the Robinson-Schensted correspondence. So ballot sequences are counted by the **involution numbers** (OEIS A000085)

$$1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, \ldots$$

with recurrence

$$t_0 = 1, \qquad t_1 = 1, \qquad \text{and} \qquad t_n = t_{n-1} + (n-1) t_{n-2} \quad \text{for} \quad n \geq 2.$$

Lazy Ballot Sequences: Voters Can Abstain

Definition

The sequence $b_1, \ldots, b_n$ where $0 \leq b_k \leq k$ is a **lazy ballot sequence** when every partial sequence $b_1, \ldots, b_k$ contains at least as many ℓ's as $(\ell+1)$'s for all $1 \leq \ell < k$.

- We allow $b_k = 0$ which corresponds to an **abstention**.
- We do not care about the (relative) number of abstentions.

examples				
0, 0, 0	0, 0, 1	0, 1, 1	0, 1, 2	1, 1, 1
	0, 1, 0	1, 0, 1	1, 0, 2	1, 1, 2
	1, 0, 0	1, 1, 0	1, 2, 0	1, 2, 1
			1, 2, 3	

Lazy Ballot Sequences Counted by Switchboard Numbers

The number of lazy ballot sequences is

$$s_n = \sum_{k=0}^{n} \binom{n}{k} t_k$$

where t_k is the number of (regular) ballot sequences of length k.

These are the **switchboard numbers** (OEIS A005425).

$$1, 2, 5, 14, 43, 142, 499, 1850, 7193, 29186, 123109, \ldots$$

which obey the recurrence

$$s_0 = 1, \qquad s_1 = 2, \qquad \text{and} \qquad s_n = 2s_{n-1} + (n-1)s_{n-2} \quad \text{for} \quad n \geq 2.$$

Approval Voting

Approval Voting

- Each voter specifies their **subset** of approved candidates.
- Each approved candidate receives one vote in their favor.
- The winner is the candidate with the most approval votes.

Example with candidates A, B and C.

Voter 1:	$\{A, B\}$
Voter 2:	$\{B, C\}$
Voter 3:	$\{B\}$
Voter 4:	$\{A, C\}$
Voter 5:	$\emptyset$
Voter 6:	$\{B, C\}$

Final Tally
A: 2
B: 4
C: 3

B is the winner.

Approval Ballot Sequence

Definition

The sequence $B_1, B_2, \ldots, B_n$ of (possibly empty) sets $B_k \subset [k]$ is an **approval ballot sequence** when for every $1 \leq \ell < k \leq n$, the partial set sequence $B_1, B_2, \ldots, B_k$ contains at least as many ℓ's as $(\ell + 1)$'s.

Example

Ballots		Partial Tallies
$B_1 =$	$\{1\}$	$(1, 0, 0)$
$B_2 =$	$\emptyset$	$(1, 0, 0)$
$B_3 =$	$\{1, 2\}$	$(2, 1, 0)$
$B_4 =$	$\{3\}$	$(2, 1, 1)$
$B_5 =$	$\{1, 2\}$	$(3, 2, 1)$
$B_6 =$	$\{2, 3\}$	$(3, 3, 2)$

partial tallies are weakly decreasing

Approval Ballot Sequences for $n = 2$

There are 7 approval ballot sequences of length 2

$\emptyset, \quad \emptyset$

$\emptyset, \quad \{1\}$

$\emptyset, \quad \{1, 2\}$

$\{1\}, \quad \emptyset$

$\{1\}, \quad \{1\}$

$\{1\}, \quad \{2\}$

$\{1\}, \quad \{1, 2\}$

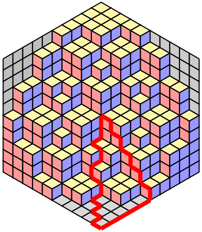
The number of approval ballot sequences of length n is

$$1, 2, 7, 42, 429, 7436, 218348, 10850216, 911835460, \ldots$$

Approval Ballot Sequences are TSSCPPs

Proposition (B, Calaway, 2022+)

Approval ballot sequences $B_1, \ldots, B_{n-1}$ are in bijection with totally symmetric self-complementary plane partitions in a $2n \times 2n \times 2n$ box.



TSSCPP

$\{1\}, \quad \{1\}, \quad \{2,3\}, \quad \{2\}$

approval ballot sequence

Approval Ballot Sequences are TSSCPPs

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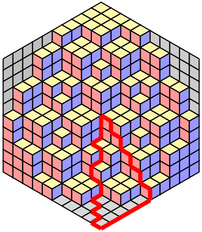

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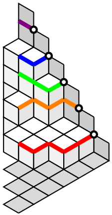
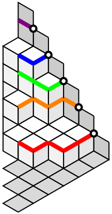
approval ballot sequence

TSSCPP to Approval Ballot Sequence

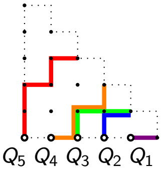


TSSCPP
fundamental domain

TSSCPP to Approval Ballot Sequence

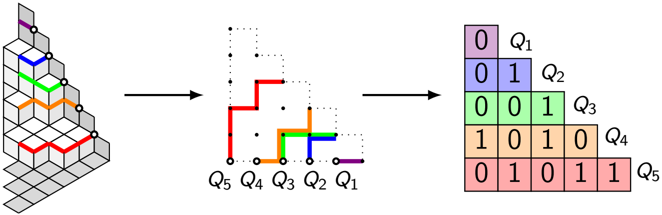


TSSCPP
fundamental domain



noncrossing
lattice paths

TSSCPP to Approval Ballot Sequence

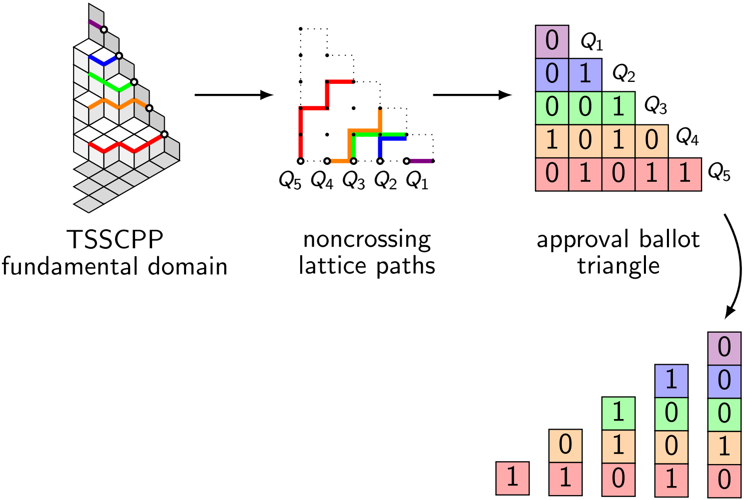


TSSCPP fundamental domain

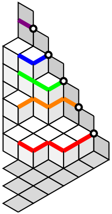
noncrossing lattice paths

approval ballot triangle

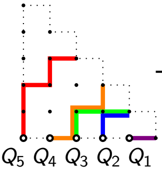
TSSCPP to Approval Ballot Sequence



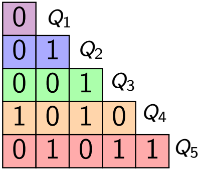
TSSCPP to Approval Ballot Sequence



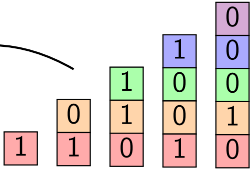
TSSCPP
fundamental domain



noncrossing
lattice paths



approval ballot
triangle



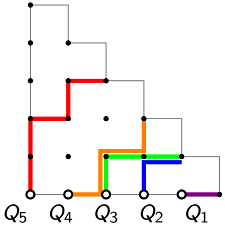
$\{1\}, \quad \{1\}, \quad \{2,3\}, \quad \{1,4\}, \quad \{2\}$

approval ballot sequence

Non-Crossing Lattice Paths

Definiton

A *nest of noncrossing lattice paths* (NCLP) of order n is a sequence of noncrossing paths $Q_1, \ldots, Q_{n-1}$ where path Q_i starts at $(n-i, 1)$ and ends at the diagonal $D = \{(n+1-j, j) : 1 \leq j \leq n\}$, taking only east $(1,0)$ steps and north $(0,1)$ steps.



NCLP of order 6

Approval Ballot Triangles

Definiton

An *approval ballot triangle* (ABT) of order n is a binary triangular array $A(i,j)$ for $1 \leq j \leq i \leq n-1$ satisfying the row compatibility condition

$$\sum_{k=j}^{i} A(i,k) \leq \sum_{k=j}^{i+1} A(i+1,k) \quad \text{for} \quad 1 \leq j \leq i \leq n-2.$$

0				
0	1			
0	0	1		
1	0	1	0	
0	1	0	1	1

ABT of order 6

Strict Sense Ballots

Strict Sense Ballot

Definition

A **strict-sense ballot** (SSB) for n candidates is a sequence of $N = \binom{n+1}{2}$ votes such that

- Candidate k receives $n+1-k$ votes
- Candidate k always leads candidate $k+1$ during the vote count.

The strict-sense ballot number sequence (OEIS A003121) begins with

$$1 \quad 1 \quad 2 \quad 12 \quad 286 \quad 33,592 \quad 23,178,480 \quad \ldots$$

and the general formula for the nth strict-sense ballot number is

$$\binom{n+1}{2}! \; \frac{\prod_{k=1}^{n-1} k!}{\prod_{k=1}^{n} (2k-1)!}.$$

Proof Part 2: Shifted SYT to Strict Sense Ballot (folklore)

Strict Sense Ballot with n candidates:

- Candidate k receives exactly $n+1-k$ votes for $1 \le k \le n$
- During the vote count, candidate k always strictly ahead of candidate ℓ for all $1 \le k < \ell \le n$

Proof Part 2: Shifted SYT to Strict Sense Ballot (folklore)

Strict Sense Ballot with n candidates:

- Candidate k receives exactly $n + 1 - k$ votes for $1 \leq k \leq n$
- During the vote count, candidate k always strictly ahead of candidate ℓ for all $1 \leq k < \ell \leq n$

Shifted SYT $\iff$ Strict Sense Ballot

1	2	3	5
	4	6	8
		7	9
			10

Proof Part 2: Shifted SYT to Strict Sense Ballot (folklore)

Strict Sense Ballot with n candidates:

- Candidate k receives exactly $n+1-k$ votes for $1 \leq k \leq n$
- During the vote count, candidate k always strictly ahead of candidate ℓ for all $1 \leq k < \ell \leq n$

Shifted SYT $\Longleftrightarrow$ Strict Sense Ballot

1	2	3	5
	4	6	8
		7	9
			10

$\Longleftrightarrow$ 1, 1, 1, 2, 1, 2, 3, 2, 3, 4

Row i gives the indices of the votes for candidate i. $\square$

Strict Sense Ballot to Sequence of NCLPs

$$(1,1,1,2,1,2,2,3,1,3,1,2,3,1,2,4,4,5,2,3,4,5,3,4,6,5,6,7)$$

strict sense ballot

1	2	3	5	9	11	14
	4	6	7	12	15	19
		8	10	13	20	23
			16	17	21	24
				18	22	26
					25	27
						28

shifted SYT

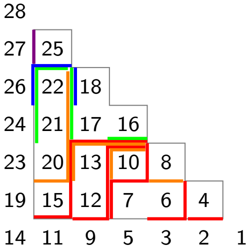
28						
27	25					
26	22	18				
24	21	17	16			
23	20	13	10	8		
19	15	12	7	6	4	
14	11	9	5	3	2	1

rotated by π

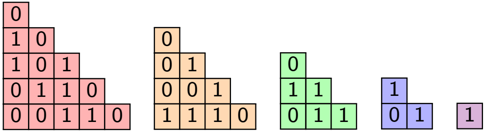
Strict Sense Ballot to Sequence of NCLPs

Theorem (B, Calaway 2022+)

Strict Sense Ballots are in bijection with Non-Crossing Lattice Paths and sequences of “compatible” ABTs.

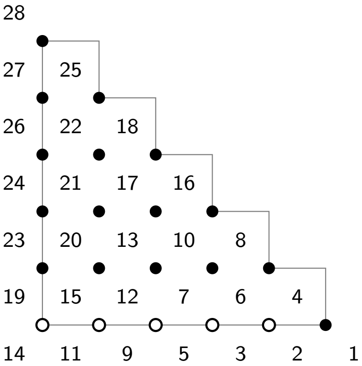


Strict Sense Ballot
& Non-Crossing Lattice Path



Sequence of Compatible ABTs

Strict Sense Ballot to Sequence of NCLPs



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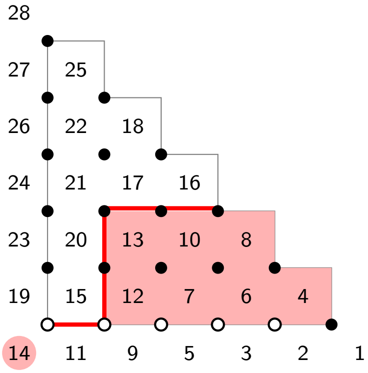

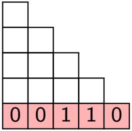

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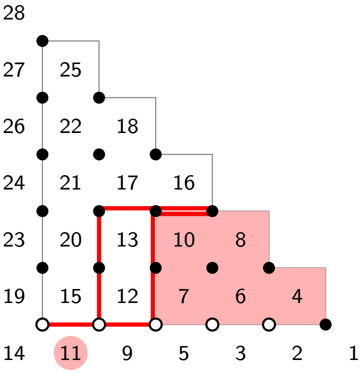

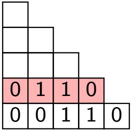

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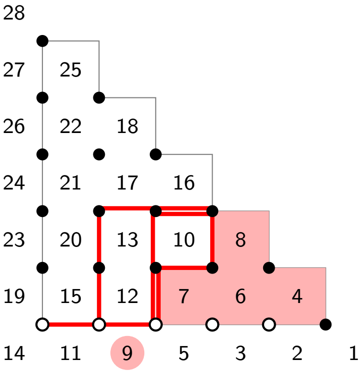

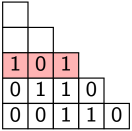

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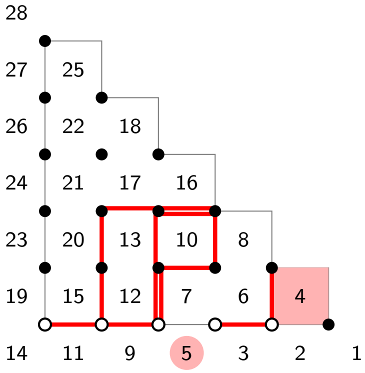
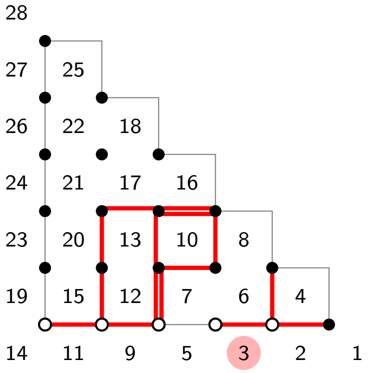


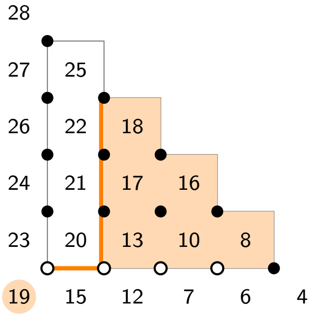
1	0			
1	0	1		
0	1	1	0	
0	0	1	1	0

Strict Sense Ballot to Sequence of NCLPs



0				
1	0			
1	0	1		
0	1	1	0	
0	0	1	1	0

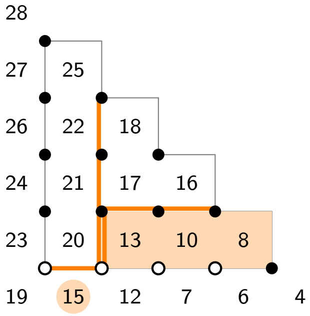
Strict Sense Ballot to Sequence of NCLPs

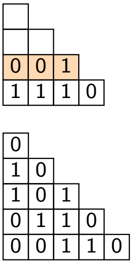


1	1	1	0

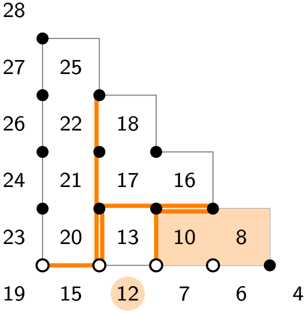
0				
1	0			
1	0	1		
0	1	1	0	
0	0	1	1	0

Strict Sense Ballot to Sequence of NCLPs





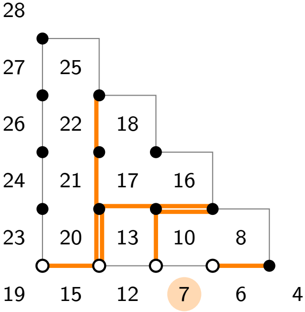
Strict Sense Ballot to Sequence of NCLPs



0	1		
0	0	1	
1	1	1	0

0				
1	0			
1	0	1		
0	1	1	0	
0	0	1	1	0

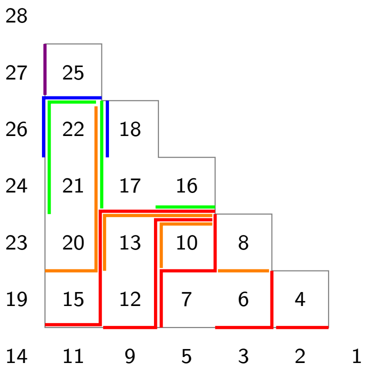
Strict Sense Ballot to Sequence of NCLPs



0			
0	1		
0	0	1	
1	1	1	0

0				
1	0			
1	0	1		
0	1	1	0	
0	0	1	1	0

Strict Sense Ballot to Sequence of NCLPs



all paths are non-crossing

1

1	
0	1

0		
1	1	
0	1	1

0			
0	1		
0	0	1	
1	1	1	0

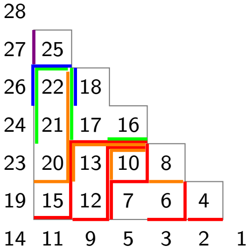
0				
1	0			
1	0	1		
0	1	1	0	
0	0	1	1	0

sequence of compatible ABTs

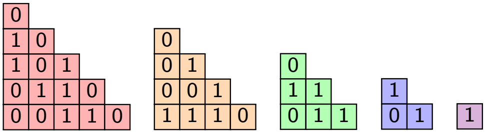
Strict Sense Ballot to Sequence of NCLPs

Theorem (B, Calaway 2022+)

Strict Sense Ballots are in bijection with Non-Crossing Lattice Paths and Sequences of Compatible ABTs, which we call Approval Ballot Hypertriangles.



Strict Sense Ballot
& Non-Crossing Lattice Path

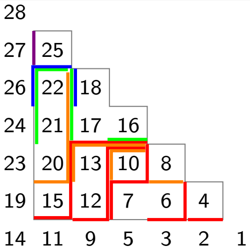


Sequence of Compatible ABTs

Strict Sense Ballot to Sequence of NCLPs

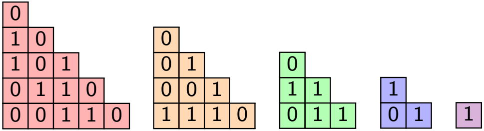
Theorem (B, Calaway 2022+)

Strict Sense Ballots are in bijection with Non-Crossing Lattice Paths and Sequences of Compatible ABTs, which we call Approval Ballot Hypertriangles.



Strict Sense Ballot

& Non-Crossing Lattice Path



~~Sequence of Compatible ABTs~~

Approval Ballot Hypertriangle

Approval Ballot Hypertriangles

Approval Ballot Hypertriangles

Definiton

An *approval ballot hypertriangle* (ABH) of order n is a sequence $(A_{n-1}, A_{n-2} \ldots, A_2)$ where A_ℓ is an ABT of order ℓ and satisfying

$$\sum_{k=j}^{i+1} A_{\ell+1}(i+1, k) \leq 1 + \sum_{k=j}^{i} A_\ell(i, k)$$

for $1 \leq j \leq i \leq \ell - 1$ and $2 \leq \ell \leq n - 2$.

$$\begin{array}{lllll}0&&&&\\1&0&&&\\1&0&1&&\\0&1&1&0&\\0&0&1&1&0\end{array}\qquad\begin{array}{llll}0&&&\\0&1&&\\0&0&1&\\1&1&1&0\end{array}\qquad\begin{array}{lll}0&&\\1&1&\\0&1&1\end{array}\qquad\begin{array}{ll}1&\\0&1\end{array}\qquad\begin{array}{l}1\end{array}$$

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0				
1	0			
1	0	1		
0	1	1	0	
0	0	1	1	0

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0				
1	0			
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0			
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0				
1	0			
1	0	1		
0	1	1	0	
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1				
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0				
1	0			
1	0	1		
0	1	1	0	
0	0	1	1	0

1				
0	1			
0	1	1		
0	0	1	1	
1	1	1	0	1

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0				
1	0			
1	0	1		
0	1	1	0	
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0			
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0		
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0				
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0	1	1	0	
0	0	1	1	0

0			
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1			
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0	1	1	0	
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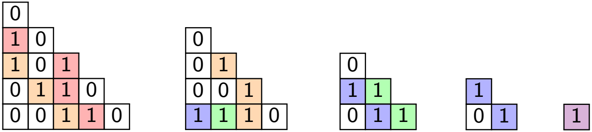
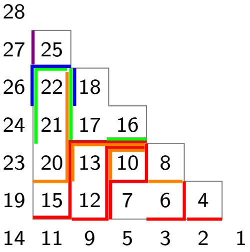


Strict Sense Ballot to Approval Ballot Hypertriangle

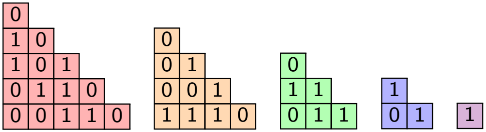
Theorem (B, Calaway 2022+)

Strict Sense Ballots of size n are in bijection with Approval Ballot Hypertriangles. of order n.



Strict Sense Ballot

candidate k gets $n+1-k$ votes



Approval Ballot Hypertriangle

sequence of approval ballots

Strict Sense Ballot to Approval Ballot Hypertriangle

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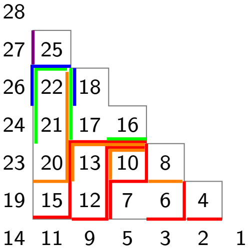
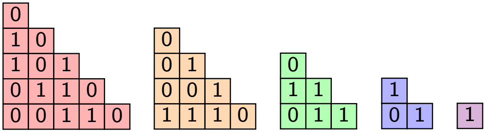


Strict Sense Ballot

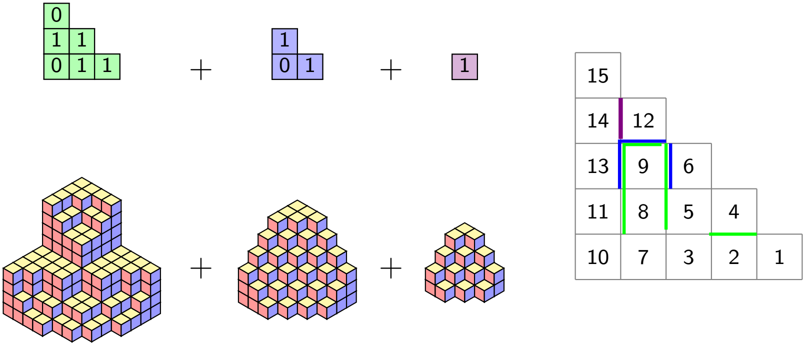


Approval Ballot Hypertriangle

candidate k gets
$n+1-k$ votes

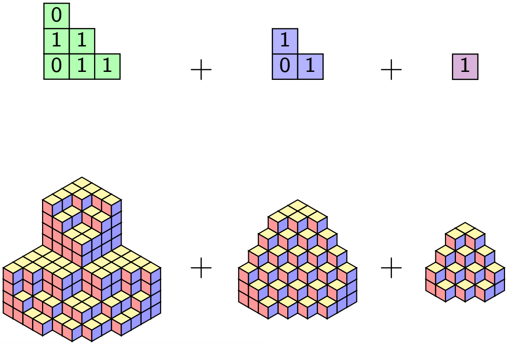
sequence of TSSCPPs!

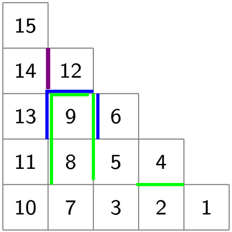
Strict Sense Ballot to Sequence of TSSCPPs



(1,1,1,2,2,3,1,2,3,1,2,4,3,4,5)

Strict Sense Ballot to Sequence of TSSCPPs





(1,1,1,2,2,3,1,2,3,1,2,4,3,4,5)

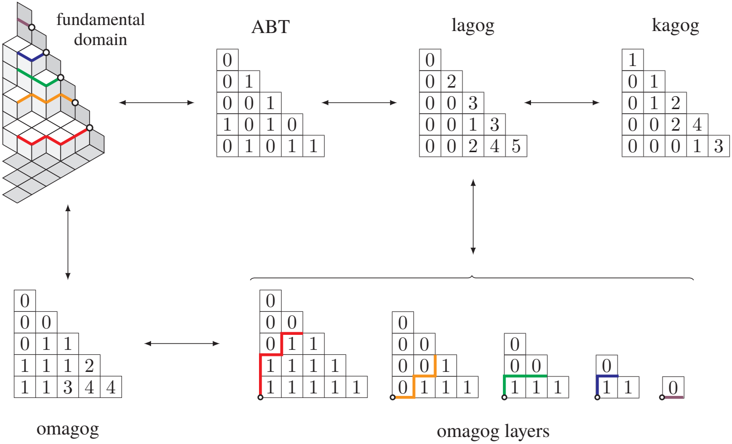
Other Triangles in Bijection with TSSCPP

We obtain more hypertriangle decompositions of a Strict Sense Ballot by using three families of triangular arrays that are in bijection with TSSCPPs.

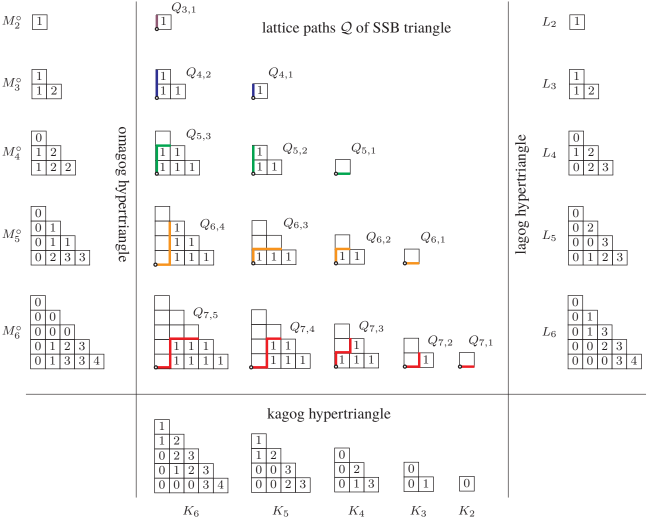
- magog triangles (Mills, Robbins, and Rumsey 1983)
- kagog triangles (B, Calaway, Heysse 2021)
- lagog triangles (new!)

Let's take a quick peek!

Triangles in Bijection with TSSCPP



Hypertriangles in Bijection with Strict Sense Ballots



Hypertriangles in Bijection with Strict Sense Ballots

Theorem (B, Calaway 2022+)

There are four different hypertriangle families that are in bijection with Strict Sense Ballots. Each of these hypertriangles encodes a sequence of compatible Totally Symmetric Self-Complementary Plane Partitions.

- A TSSCPP $\longleftrightarrow$ list of compatible lattice paths of decreasing sizes
- An SSB $\longleftrightarrow$ list of compatible TSSCPP of decreasing sizes

Hypertriangles in Bijection with Strict Sense Ballots

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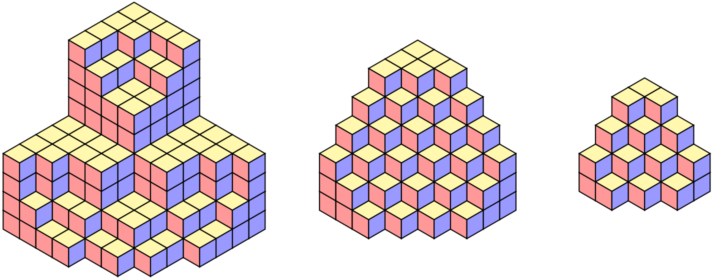
- A TSSCPP $\longleftrightarrow$ list of compatible lattice paths of decreasing sizes
- An SSB $\longleftrightarrow$ list of compatible TSSCPP of decreasing sizes
- An ??? $\longleftrightarrow$ list of compatible SSBs of decreasing sizes

A Natural Law of Combinatorics

Careful aggregation of combinatorial structures gives rise to other interesting combinatorial structures



$$= (1,1,1,2,2,3,1,2,3,1,2,4,3,4,5)$$